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Any algebraic variety in positive characteristic admits a projective model with an inseparable Gauss map

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ABSTRACT

We determine the values attained by the rank of the Gauss map of a projective model for a fixed algebraic variety in positive characteristic p. In particular, it is shown that any variety in $p > 0$ has a projective model such that the differential of the Gauss map is identically zero. On the other hand, we prove that there exists a product of two or more projective spaces admitting an embedding into a projective space such that the differential of the Gauss map is identically zero if and only if $p = 2$.

1. Introduction

Let X be a projective variety in $\mathbb{P}^N$ defined over an algebraically closed field K of characteristic $p \geq 0$. The *Gauss map* of $X \subseteq \mathbb{P}^N$, denoted by γ, is by definition the rational map from X to the Grassmann variety $\mathbb{G}(n, \mathbb{P}^N)$ which sends each smooth point P of X to the embedded tangent space $T_P X$ to X at P in $\mathbb{P}^N$ ([1, Section 1, (e)], [2, I, Section 2]), where we set $n := \dim X$.

In characteristic zero, it is classically known that a general fiber of the Gauss map γ is a linear space ([1, (2.10)], [2, I, 2.3. Theorem]); hence, if γ is generically finite, for example if X is smooth [2, I, 2.8. Corollary], then γ is birational onto its image. In positive characteristic, this is no longer true, and various pathological phenomena concerning the behavior of embedded tangent spaces have been observed by several authors. Those phenomena seem to be caused by the inseparability of Gauss maps [3–9]. To analyze those pathological phenomena it would be important to consider the following

Question 1.1. *What projective variety has an embedding into a projective space with an inseparable Gauss map?*

Our answer here is

Theorem 1.2. *For a projective variety X of dimension n in $p > 0$ and an integer r with $0 \leq r \leq n$, the following conditions are equivalent:*

(1) *there exists a birational embedding of X into a projective space such that the Gauss map γ is generically finite and the differential of γ has rank r;*
(2) *$r \neq 1$ if $p = 2$.*

Moreover, for any r satisfying (2), there exists a birational embedding of X such that the extension $K(X)/K(\gamma(X))$ of function fields defined by γ is purely inseparable of degree p^{n-r}.

A *birational embedding* here means a rational map which is birational onto the closure of its image, while a *biregular embedding* below means a morphism which is isomorphic onto its image. The result above asserts especially that any variety in $p > 0$ has a projective model such that γ is generically finite and the differential of γ is identically zero. Note that this is

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already known for any one-dimensional X ([5, Corollary 3.4], [10, I-3]). Note also that, to obtain the conclusion, it is necessary to consider not only biregular embeddings, but also birational embeddings in Theorem 1.2. In fact, we have

Theorem 1.3. *There exists a product $\prod_{1\le i\le r}\mathbb{P}^{n_i}$ of two or more projective spaces ($r \ge 2, n_i \ge 1$) admitting a biregular embedding such that the differential of the Gauss map is identically zero if and only if $p = 2$.*

This paper is organized as follows. In Section 2 we discuss the differential of the Gauss map and treat basic facts on it: in particular, we give a criterion for the Gauss map to have differential identically zero. We prove Theorems 1.2 and 1.3 respectively in Sections 3 and 4.

This is a modified version of an unpublished paper [11] by the authors.

2. The differential of the Gauss map

Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n, and let P be a smooth point of X. One may assume that, in terms of homogeneous coordinates, the embedding $\rho : X \hookrightarrow \mathbb{P}^N$ is given locally by

$$\rho = (1 : x_1 : \cdots : x_n : f_{n+1} : \cdots : f_N), \tag{2.1}$$

where $x_1, \ldots, x_n$ form a regular system of parameters of the local ring $\mathcal{O}_{X,P}$ and $f_k \in \mathcal{O}_{X,P}$. The reason is as follows, by an elementary argument. For simplicity, one may assume that $P = (1 : 0 : \cdots : 0)$. Then, the coordinate functions $x_1, \ldots, x_N$ are contained in and generate the maximal ideal m_P of $\mathcal{O}_{X,P}$; hence the residue classes $\bar{x_1}, \ldots, \bar{x_N}$ span the cotangent space m_P/m_P^2. Since $\dim m_P/m_P^2 = n$ by the smoothness of P, one may assume that $\bar{x_1}, \ldots, \bar{x_n}$ form a basis for m_P/m_P^2. Then, $x_1, \ldots, x_n$ generate m_P by Nakayama's lemma, and ρ is written as above with $f_k := x_k$ for $n + 1 \le k \le N$.

Since the embedded tangent space T_PX to X at P is spanned by $\rho(P), \rho_{x_1}(P), \ldots, \rho_{x_n}(P)$, the Gauss map γ is given by an $(n + 1) \times (N + 1)$-matrix,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & f_k - \sum_{j=1}^{n} x_j f_{k,x_j} & \cdots \\ 0 & 1 & \cdots & 0 & \cdots & f_{k,x_1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & \cdots & f_{k,x_n} & \cdots \end{bmatrix}. \tag{2.2}$$

Therefore the rank of the differential of γ is equal to that of an $n \times (N - n)n$-matrix consisting of Hessian matrices as follows (see [12, Section 2, (2)]):

$$\operatorname{rk} \mathrm{d}\gamma = \operatorname{rk}[H(f_{n+1}) \cdots H(f_N)], \tag{2.3}$$

where $H(f) := [f_{x_i x_j}]_{1\le i,j\le n}$ is the Hessian matrix of f. By $\mathrm{d}\gamma \equiv 0$ we mean that the differential of the Gauss map γ is identically zero. Then we have the following

Lemma 2.1. *The differential $\mathrm{d}\gamma \equiv 0$ if and only if the functions $f_{k,x_i x_j} \in K(X)$ are equal to zero for all i, j, k.*

Hereinafter in this section we assume that $X \subset \mathbb{P}^N$ is smooth, for simplicity. Let $T(X)$ be the affine tangent bundle on X:

$$T(X) := \{(P, y) \in X \times \mathbb{A}^{N+1} | y \in \hat{T}_P X\},$$

and let $\theta : T(X) \to \mathbb{A}^{N+1}$ be the second projection, where $\hat{T}_P X \subset \mathbb{A}^{N+1}$ denotes the affine cone of $T_P X \subset \mathbb{P}^N$.

Lemma 2.2. *The differential $\mathrm{d}\gamma \equiv 0$ if and only if $\mathrm{d}\theta$ has rank $n + 1$ at any point $(P, y) \in T(X)$.*

Proof. Since $\theta : \operatorname{Spec} \mathcal{O}_{X,P} \times \mathbb{A}^{n+1} \to \mathbb{A}^{N+1}$ is given by $(x) \times (t_0, \ldots, t_n) \mapsto t_0\rho + t_1\rho_{x_1} + \cdots + t_n\rho_{x_n}$, the image of $d\theta$ is spanned by

$$\rho, \rho_{x_1}, \ldots, \rho_{x_n}, \sum_j t_j \rho_{x_1 x_j}, \ldots, \sum_j t_j \rho_{x_n x_j}.$$

From Lemma 2.1 we find that $\mathrm{d}\gamma \equiv 0$ if and only if the image of $\mathrm{d}\theta$ has dimension $n + 1$. □

Let $L \subset \mathbb{P}^N$ be a linear subspace, and denote by $\pi_L : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N'}$ a projection from L. We denote by $\hat{L} \subset \mathbb{A}^{N+1}$ the affine cone of L, and by $\hat{\pi}_L : \mathbb{A}^{N+1} \to \mathbb{A}^{N'+1}$ the projection from $\hat{L}$. To be more specific, we write γ_X for γ, θ_X for θ, and so on.

Lemma 2.3. *Assume that $L \cap X = \emptyset$ and π_L gives an isomorphism from X onto its image X'. Then $\mathrm{d}\gamma_{X'} \equiv 0$ if and only if $\rho_{x_i x_j} \in \hat{L}$ for all i, j.*

Proof. Since we have a commutative diagram:

$$\begin{array}{ccc} T(X) & \xrightarrow{\theta_X} & \mathbb{A}^{N+1} \\ \downarrow \wr & & \downarrow \hat{\pi}_L \\ T(X') & \xrightarrow{\theta_{X'}} & \mathbb{A}^{N'+1}, \end{array}$$

it follows from Lemma 2.2 that $d\gamma_{X'} \equiv 0$ if and only if $d(\hat{\pi}_L \circ \theta_X)$ has rank $n+1$, which is equivalent to saying that $\sum_j t_j \rho_{x_1 x_j}, \ldots, \sum_j t_j \rho_{x_n x_j} \in \hat{L}$ for any $(t_0, \ldots, t_n) \in \mathbb{A}^{n+1}$. This implies our assertion. □

3. The rank of the Gauss map

For a projective variety X in $\mathbb{P}^N$, taking a suitable system of homogeneous coordinates for $\mathbb{P}^N$, one may assume that the embedding $\rho : X \hookrightarrow \mathbb{P}^N$ is given locally as in (2.1).

Proof of Theorem 1.2. (1) $\Rightarrow$ (2). Suppose that the Gauss map γ of ρ had differential of rank 1 in $p = 2$. Then the rank of $H(f_k)$ would be equal to 1 for some k by (2.3). On the other hand, since the diagonal elements of $H(f_k)$ are all zero by $p = 2$, $H(f_k)$ is symmetric as well as skew-symmetric again by $p = 2$; hence the rank of $H(f_k)$ must be even [13, Section 5, $n^\circ 1$, Corollaire 3]. This is a contradiction.

(2) $\Rightarrow$ (1). Let r be an integer with $0 \le r \le n$, and assume $r \ne 1$ if $p = 2$. Denote by $\Omega_{K(X)/K}$ the module of differentials of $K(X)$ over K. Since the Kähler differentials $dx_1, \ldots, dx_n$ form a basis of a vector space $\Omega_{K(X)/K}$ over $K(X)$, according to [14, Theorem 26.5], the x_i form a p-basis of $K(X)/K$; that is, the set $\{x_1^{i_1} \cdots x_n^{i_n}\}_{0 \le i_1, \ldots, i_n < p}$ of p-monomials is a basis for a vector space $K(X) = K(X)^p(x_1, \ldots, x_n)$ over $K(X)^p$. Therefore each f_k above is written uniquely as

$$f_k = \sum_{0 \le i_1, \ldots, i_n < p} g(k; i_1, \ldots, i_n) x_1^{i_1} \cdots x_n^{i_n}, \quad \text{with } g(k; i_1, \ldots, i_n) \in K(X)^p. \tag{3.1}$$

We now define a rational map $\tau : X \dashrightarrow \mathbb{P}^M$ by

$$\tau = \left(1 : x_1 : \cdots : x_n : \cdots : x_1 x_i : \cdots : x_j^p : \cdots : f_k^p : \cdots : g(l; i_1, \ldots, i_n) : \cdots \right),$$

where $M = n + N + (N-n)p^n$, the indices are running in the range $1 \le i \le r < j \le n < k, l \le N, 0 \le i_1, \ldots, i_n < p$, and the $x_1 x_i$ terms do not appear if $r = 0$. Then τ is birational onto its image. Indeed, if Y denotes the closure of the image $\tau(X)$, then $f_k \in K(Y)$ by (3.1); hence $K(X) = K(Y)$. Let γ be the Gauss map of τ. We see from (2.2) that the functions x_i, x_j^p, f_k^p are contained in $K(\gamma(X))$; hence $K(X)^p \subseteq K(\gamma(X)) \subseteq K(X)$. In particular, γ is generically finite. Moreover, by $g(l; i_1, \ldots, i_n) \in K(X)^p$, it follows that $K(\gamma(X)) = K(X)^p(x_1, \ldots, x_r)$. Since the x_i form a p-basis of $K(X)/K$, $K(X)^p(x_1, \ldots, x_r)$ has dimension p^r over $K(X)^p$; hence $K(X)/K(\gamma(X))$ is a finite, purely inseparable extension of degree p^{n-r}. From a fundamental exact sequence of vector spaces over $K(X)$ [14, Theorem 25.1]:

$$\Omega_{K(\gamma(X))/K} \otimes_{K(\gamma(X))} K(X) \xrightarrow{\gamma^*} \Omega_{K(X)/K} \longrightarrow \Omega_{K(X)/K(\gamma(X))} \longrightarrow 0,$$

we see that $\operatorname{rk} \gamma^* = n - \dim \Omega_{K(X)/K(\gamma(X))}$, and $\dim \Omega_{K(X)/K(\gamma(X))} = n - r$ [15, Chapter II, Section 17, Corollary 5]); hence we have $\operatorname{rk} d\gamma = \operatorname{rk} \gamma^* = r$, as is required. □

Remark 3.1. The generic finiteness of γ is not necessary for the proof of (1) $\Rightarrow$ (2) above.

Remark 3.2. In the last part of the proof of (2) $\Rightarrow$ (1) above, one can calculate the rank of $d\gamma$ directly as follows. According to (2.3), we have

$$\operatorname{rk} d\gamma = \operatorname{rk}[\cdots H(x_1 x_i) \cdots H(x_j^p) \cdots H(f_k^p) \cdots H(g(l; i_1, \ldots, i_n)) \cdots].$$

Since $H(x_j^p) = H(f_k^p) = H(g(l; i_1, \ldots, i_n)) = 0$ because of the p-th power, it follows that $\operatorname{rk} d\gamma = \operatorname{rk}[\cdots H(x_1 x_j) \cdots]_{1 \le j \le r}$, which is equal to r.

Remark 3.3. According to [16, Theorem 1.1], if the Gauss map of a projective variety X of dimension at least 3 has differential identically zero, then X has a non-reflexive embedding with a birational Gauss map. Thus, combining with Theorem 1.2, one can deduce immediately that *any algebraic variety of dimension at least* 3 *in* $p > 0$ *has a non-reflexive projective model with a birational Gauss map.* This consequence tells that there are so many examples of projective varieties which answer a question [17, pp. 108–109] raised by S. L. Kleiman and R. Piene in the negative.

4. Embeddings of a product of projective spaces

Proof of Theorem 1.3. Suppose that $X = \prod_{1 \le i \le r} \mathbb{P}^{n_i}$ had such a biregular embedding and $p > 2$. One may assume that the embedding in question is a composition of a morphism $\rho : X \hookrightarrow \mathbb{P}^N$ defined by some complete linear system with a projection π_L from some linear subspace $L \subset \mathbb{P}^N$. Denote by X' the isomorphic image of X under $\pi_L \circ \rho$, and by $\gamma_{X'}$ the Gauss map of X' as in Section 2.

Firstly, we consider the case $n_1 = n_2 = 1$ and $r = 2$, that is, $X = \mathbb{P}^1 \times \mathbb{P}^1$, and let (a, b) be the type of $\rho^* \mathcal{O}_{\mathbb{P}^N}(1)$ via $\operatorname{Pic} X \simeq \mathbb{Z} \oplus \mathbb{Z}$. One may assume that ρ is given by $(1 : x) \times (1 : y) \mapsto (1 : x : y : xy : \cdots : x^a y^b)$. If $d\gamma_{X'}$ were identically zero, then we would have $\rho_{xx}(x, y), \rho_{xy}(x, y), \rho_{yy}(x, y) \in \hat{L}$ by Lemma 2.3. Now, let $\phi : \mathbb{P}^1 \to X$ be a morphism defined by $(1 : t) \mapsto (1 : \lambda) \times (1 : t)$ with $\lambda \in K$, and set $\phi^L := \pi_L \circ \rho \circ \phi$. Since

$$\rho \circ \phi(t) = (1 : \lambda : t : \lambda t : \cdots : \cdots : \lambda^a t^b),$$

we have $(\rho \circ \phi(t))_{tt} = \rho_{xx}(\lambda, t) \in \hat{L}$; hence the Gauss map of $\phi^L(\mathbb{P}^1)$ is inseparable by Lemma 2.3. Therefore, by virtue of the Plücker formula [18, Section 3] we have $b \equiv 1 \bmod p$, because $p > 2$ and $\phi^L(\mathbb{P}^1)$ has degree b. Similarly, we have $a \equiv 1 \bmod p$. We next consider the diagonal morphism $\Delta : \mathbb{P}^1 \to X$, and set $\Delta^L := \pi_L \circ \rho \circ \Delta$. Then we have

$$\rho \circ \Delta(t) = (1 : t : t : t^2 : \cdots : t^{a+b});$$

hence $(\rho \circ \Delta(t))_{tt} = \rho_{xx}(t, t) + 2\rho_{xy}(t, t) + \rho_{yy}(t, t) \in \hat{L}$. Therefore, similarly to the above, the Gauss map of $\Delta^L(\mathbb{P}^1)$ is inseparable by Lemma 2.3, and $a + b \equiv 1 \bmod p$ by the Plücker formula, where $\Delta^L(\mathbb{P}^1)$ has degree $a + b$. This contradicts $a \equiv b \equiv 1 \bmod p$.

Now, consider a general case. Let $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to X$ be a morphism defined by $(1 : x) \times (1 : y) \mapsto (1 : x : 0 : \cdots : 0) \times (1 : y : 0 : \cdots : 0) \times \mathrm{pt}$. If $d\gamma_{X'}$ were identically zero, then we would have $(\rho \circ \phi(x, y))_{xx}, (\rho \circ \phi(x, y))_{xy}, (\rho \circ \phi(x, y))_{yy} \in \hat{L}$ by a similar computation as above; hence the differential of the Gauss map of $\pi_L \circ \rho \circ \phi(\mathbb{P}^1 \times \mathbb{P}^1)$ would be identically zero. However, $\mathbb{P}^1 \times \mathbb{P}^1$ has no such embedding, as we have just proven.

To prove the converse, assume that $p = 2$. Then a product $\mathbb{P}^1 \times \mathbb{P}^1$ has a biregular embedding with $d\gamma \equiv 0$. In fact, we see from (2.3) that an embedding $\rho : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7$ defined by the following satisfies the required property:

$$\rho((1 : x) \times (1 : y)) = (1 : x : y : x^2 : y^2 : xy^2 : x^2y : x^2y^2). \quad \square$$

Remark 4.1. Let $Y \subset X \subset \mathbb{P}^N$ be projective varieties with $\dim Y = r < \dim X = n$. We note that $d\gamma_X \equiv 0$ does not imply $d\gamma_Y \equiv 0$ in general. For example, let $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of the Frobenius morphism of $\mathbb{P}^2$, $X \subset \mathbb{P}^8$ the image of Γ under the Segre embedding, and $Y \subset X$ the image of a conic in $\mathbb{P}^2 \simeq X$. Then it is easily verified by a direct computation that $d\gamma_X \equiv 0$, but $d\gamma_Y \not\equiv 0$ if $p \neq 2$.

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